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# Fat Homeomorphisms and Unbounded Derivate Containers

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We generalize local and global inverse function theorems to continuous transformations in $\mathbb{R}^n$, replacing nonexistent derivatives by set-valued "unbounded derivate containers." We also construct and study unbounded and ordinary derivate containers, including extensions of generalized Jacobians.

## 1. INTRODUCTION

Let $B(x, a)$ $[\bar{B}(x, a)]$ denote the open [closed] ball of center $x$ and radius $a$. The classical inverse function theorem implies that a $C^1$ function $f$ in a Banach space whose derivative $f'(\bar{x})$ has a continuous inverse is *fat at* $\bar{x}$, i.e., there exists $c > 0$ such that

$$f(\bar{B}(\bar{x}, a)) \supset \bar{B}(f(\bar{x}), ca)$$

for sufficiently small positive $a$. A global version of the inverse function theorem [7, Lemma 1, p. 66] asserts that if $f$ is $C^1$ in some neighborhood $X$ of $\bar{B}(\bar{x}, \alpha)$ and $|f'(x)^{-1}| \leqslant \beta$ $(x \in X)$ then

$$f(\bar{B}(\bar{x}, a)) \supset \bar{B}(f(\bar{x}), a/\beta) \qquad (0 \leqslant a \leqslant \alpha)$$

and a restriction of $f$ has a $C^1$ inverse $u\colon \bar{B}(f(\bar{x}), \alpha/\beta) \to \bar{B}(\bar{x}, \alpha)$. If $f$ is a Lipschitzian mapping in $\mathbb{R}^n$ then a known result concerning functions defined on convex sets [5, Lemma 3.3, p. 554] yields, as special cases, fat mapping and local and global homeomorphism theorems formulated in terms of derivate containers $\Lambda f(x)$. The latter play the role of set-valued derivatives and are, crudely speaking, sets of limits of $f_i'(y)$ as $i \to \infty$ and $y \to x$, where $(f_i)$ is any sequence of $C^1$ functions converging uniformly to $f$. In these results the open mapping property follows from the nonsingularity of the

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elements of $\Lambda f(x)$ and the existence of a local (global) inverse follows from the nonsingularity of the elements of the convex closure of $\Lambda f(x)$ $(\bigcup_x \Lambda f(x))$.

In the present paper we introduce *unbounded derivate containers* which are generalizations of derivate containers applicable to continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ that are not necessarily Lipschitzian. Our purpose is twofold: (a) to establish fat homeomorphism theorems without assumptions of Lipschitz continuity of $f$ or of convexity of its ordinary or unbounded derivate containers, and (b) to study and compare certain types of convex and nonconvex unbounded derivate containers. We shall prove, in particular, that Clarke's generalized Jacobians [2] and certain extensions to functions with integrably bounded finite difference quotients are the best (i.e., the smallest) unbounded derivate containers among those that are closed and convex. This will generalize a prior result [6, Theorem 2.10, p. 17] that held for Lipschitzian functions with either their domain or their range in $\mathbb{R}$. However, we shall also give an example of a Lipschitzian function with a "singular" generalized Jacobian but a "nonsingular" nonconvex derivate container. This example shows that nonconvex derivate containers are not only a stronger tool for the study of the minima of functions [6, 2.13, pp. 17, 18] but also for the study of the invertibility and open mapping properties.

We ought to mention that the set-valued $D$-derivative of Mordukhovich [4], defined for scalar-valued functions only, may have some advantages over ordinary or unbounded derivate containers in the study of minima.

Our main results are presented in Section 2. Section 3 is devoted to some remarks, examples, and open questions. The proofs are contained in Section 4.

## 2. Main Results

Let $n$ and $m$ be positive integers and $|\cdot|$ denote arbitrary norms in $\mathbb{R}^n$ and $\mathbb{R}^m$ and the corresponding norm in $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$. If $M \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$ is a singular matrix then we write $|M^{-1}| = \infty$. Let $V \subset \mathbb{R}^n$ be open and $f: V \to \mathbb{R}^m$ continuous.

Definition A. We refer to a collection $\{\Lambda^\varepsilon f(x) \mid \varepsilon > 0,\ x \in V\}$ of nonempty subsets of $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$, also denoted by $\Lambda^\varepsilon f$, as an unbounded derivate container for $f$ if

$$\Lambda^\varepsilon f(x) \subset \Lambda^{\varepsilon'} f(x) \qquad (\varepsilon' > \varepsilon)$$

and for every compact $V^* \subset V$ there exist a sequence $(f_i)$ of $C^1$ functions defined in some neighborhood of $V^*$ and numbers $i(\varepsilon, V^*)$, $\delta(\varepsilon, V^*) > 0$ $(\varepsilon > 0)$ such that $\lim_i f_i = f$ uniformly on $V^*$ and

$$f_i'(y) \in \Lambda^\varepsilon f(x) \qquad (i \geqslant i(\varepsilon, V^*),\ x \in V^*, y \in V, |x-y| \leqslant \delta(\varepsilon, V^*))$$

If the sets $\Lambda^\varepsilon f(x)$ ($\varepsilon > 0$, $x \in V$) are all closed and uniformly bounded then the unbounded derivate container $\Lambda^\varepsilon f$ is a derivate container (as defined in [5, 2.1, p. 549]). We then write

$$\Lambda f(x) = \bigcap_{\varepsilon > 0} \Lambda^\varepsilon f(x).$$

THEOREM 1. *Let $n = m$ and $\Lambda^\varepsilon f$ be an unbounded derivate container for $f$. Assume that $\alpha > 0$, $\bar{B}(\bar{x}, \alpha) \subset V$, and*

$$\beta = \sup_{x \in \bar{B}(\bar{x}, \alpha)} \inf_{\varepsilon > 0} \sup\{|M^{-1}| \mid M \in \Lambda^\varepsilon f(x)\} < \infty.$$

*Then*

$$f(\bar{B}(\bar{x}, a)) \supset \bar{B}(f(\bar{x}), a/\beta) \qquad (0 \leqslant a \leqslant \alpha)$$

*and there exists an open $A \subset \bar{B}(\bar{x}, \alpha)$ such that $f: \bar{A} \to \bar{B}(f(\bar{x}), \alpha/\beta)$ is a homeomorphism and its inverse has a Lipschitz constant $\beta$.*

*If $\Lambda^\varepsilon f$ is a derivate container for $f$ then the above conclusion holds with $\beta$ replaced by*

$$\gamma = \sup\{|M^{-1}| \mid M \in \Lambda f(x), x \in \bar{B}(\bar{x}, \alpha)\}.$$

THEOREM 2. *Let $n = m$, $\Lambda^\varepsilon f$ be an unbounded derivate container for $f$ and*

$$\beta_0 = \inf_{\varepsilon > 0} \sup\{|M^{-1}| \mid M \in \Lambda^\varepsilon f(\bar{x})\} < \infty.$$

*Then for each $\beta > \beta_0$ there exist $\alpha = \alpha_\beta > 0$ and an open $A = A_\beta \subset \bar{B}(\bar{x}, \alpha)$ such that*

$$f(\bar{B}(\bar{x}, a)) \supset \bar{B}(f(\bar{x}), a/\beta) \qquad (0 \leqslant a \leqslant \alpha)$$

*and $f: \bar{A} \to \bar{B}(f(\bar{x}), \alpha/\beta)$ is a homeomorphism.*

*If $\Lambda^\varepsilon f$ is a derivate container for $f$ then the above conclusion holds with $\beta_0$ replaced by*

$$\gamma_0 = \sup\{|M^{-1}| \mid M \in \Lambda f(\bar{x})\}.$$

We next consider the problem of constructing particular unbounded derivate containers. We observe that a particular unbounded derivate container for $f$ can be defined by the collection

$$\Phi^\varepsilon f(x) = \{f_i'(y) \mid y \in V, |y - x| \leqslant \varepsilon, i \geqslant 1/\varepsilon\} \qquad (\varepsilon > 0, x \in V)$$

if $(f_i)$ is a sequence of $C^1$ functions such that, for each compact $V^*$, $f_i$ is defined in some neighborhood of $V^*$ for sufficiently large $i$ and $\lim_i f_i = f$ uniformly on $V^*$. Furthermore, if

$$\Omega^\varepsilon f(x) \supset \Phi^\varepsilon f(x) \text{ and } \Omega^\varepsilon f(x) \subset \Omega^{\varepsilon'} f(x) \qquad (\varepsilon' > \varepsilon,\ x \in V)$$

then clearly $\Omega^\varepsilon f$ is also an unbounded derivate container for $f$. Finally, if an unbounded derivate container $\Lambda^\varepsilon f$ for $f$ is such that the sets $\Lambda^\varepsilon f(x)$ $(\varepsilon > 0,$ $x \in V)$ are uniformly bounded then the sets

$$\Lambda_1^\varepsilon f(x) = \text{closure}\, \Lambda^\varepsilon f(x)$$

define a derivate container for $f$.

The above considerations show that a function $f: V \to \mathbb{R}^m$ has an unbounded derivate container, respectively, a derivate container if and only if it is continuous, respectively, Lipschitzian. The necessity is obvious because uniform limits of $C^1$ functions are continuous, and they are Lipschitzian if the approximating functions have uniformly bounded derivatives. The sufficiency follows from the fact that a continuous, respectively, Lipschitzian function is appropriately approximated by functions $f_i$ which can be generated by convolutions of $f$ with nonnegative $C^1$ "$\delta$-function approximations."

Because we shall be able to derive useful properties of the corresponding unbounded derivate containers, we shall apply a somewhat more general approximation procedure to a special family of functions. We shall denote by $\mathscr{F}$ the collection of all continuous functions $g: V \to \mathbb{R}^m$ for which there exist a locally integrable $\psi_g: V \to \bar{\mathbb{R}}$ and $a_g > 0$ such that

$$|y-x|^{-1}\, |g(y) - g(x)| \leqslant \psi_g(x) \qquad (x, y \in V, 0 < |y-x| \leqslant a_g).$$

It follows from Stepanoff's theorem [3, 3.1.9, p. 218] that each $g \in \mathscr{F}$ is differentiable a.e. in $V$.

For simplicity of notation we extend each $g \in \mathscr{F}$ to all of $\mathbb{R}^n$ by setting $g(x) = 0$ $(x \notin V)$. We write $\int h(z)\, dz$ for the integral with respect to the $n$-dimensional Lebesgue measure which we denote by meas. The symbols $|\cdot|_\infty$ and $|\cdot|_1$ represent the $L^\infty$(meas) and $L^1$(meas) norms. We also write $\bar{A}$ or $\operatorname{cl} A$ for the closure of $A$, co for the convex hull and $\overline{\text{co}}$ for the convex closure.

THEOREM 3. *Let $f \in \mathscr{F}$ and $p_i: \mathbb{R}^n \to \mathbb{R}$ $(i = 1, 2, \ldots)$ be bounded measurable functions such that*

$$p_i(x) = 0 \qquad (|x| > 1/i), \int p_i(x)\, dx = 1, c = \sup_i |p_i|_1 < \infty.$$

*Set*

$$f_i(x) = \int f(x+z)\, p_i(z)\, dz \qquad (i = 1, 2, \ldots, \bar{B}(x, 1/i) \subset V).$$

*Then each* $f_i$ *is* $C^1$, $\lim_i f_i = f$ *uniformly on every compact subset of* $V$, *and*

$$f_i'(x) = \int f'(x+z)\, p_i(z)\, dz \qquad (i = 1, 2, \ldots, \bar{B}(x, 1/i) \subset V).$$

*Furthermore, the sets*

$$\Lambda^\varepsilon f(x) = \{ f_i'(y) \mid |y - x| \leqslant \varepsilon, \bar{B}(y, 1/i) \subset V, i \geqslant 1/\varepsilon \}$$

*define an unbounded derivate container for* $f$.

We finally consider an extension of Clarke's generalized Jacobians [2] and a chain rule for unbounded derivate containers. For any $g \in \mathscr{F}$ and any $N \subset \mathbb{R}^n$, we set

$$V_g = \{ y \in V \mid g'(y) \text{ exists} \}$$

$$\partial_N^\varepsilon g(x) = \overline{\text{co}} \{ g'(y) \mid |y - x| \leqslant \varepsilon, y \in V_g \sim N \} \qquad (x \in V, \varepsilon > 0)$$

$$\partial_N g(x) = \bigcap_{\varepsilon > 0} \partial_N^\varepsilon g(x), \qquad \partial^\varepsilon g(x) = \partial_\varnothing^\varepsilon g(x), \partial g(x) = \partial_\varnothing g(x)$$

THEOREM 4. *Let* $f \in \mathscr{F}$ *and* $\text{meas}(N) = 0$. *Then*

$$\partial^\varepsilon f(x) \subset \partial_N^{\varepsilon'} f(x) \subset \partial^{\varepsilon'} f(x) \qquad (\varepsilon' > \varepsilon > 0, x \in V),$$

*and* $\partial_N^\varepsilon f$ *is an unbounded derivate container for* $f$. *If* $\Lambda^\varepsilon f$ *is any other unbounded derivate container for* $f$ *then for each compact* $V^* \subset V$ *there exist numbers* $\delta(\varepsilon) > 0$ $(\varepsilon > 0)$ *such that*

$$\partial^{\delta(\varepsilon)} f(x) \subset \overline{\text{co}}\, \Lambda^\varepsilon f(x) \qquad (x \in V^*, \varepsilon > 0).$$

*If* $\Lambda^\varepsilon f$ *is a derivate container for* $f$ *then we also have*

$$\partial_N f(x) = \partial f(x) \subset \text{co}\, \Lambda f(x) \qquad (x \in V).$$

THEOREM 5. *Let* $V_1 \subset \mathbb{R}^{n_1}$ *and* $V_2 \subset \mathbb{R}^{n_2}$ *be open and* $g: V_1 \to V_2$ *and* $h: V_2 \to \mathbb{R}^m$ *continuous with unbounded derivate containers* $\Lambda^\varepsilon g$ *and* $\Lambda^\varepsilon h$. *Then the sets*

$$\Lambda^\varepsilon \phi(x) = \Lambda^\varepsilon h(g(x))\, \Lambda^\varepsilon g(x) = \{ MN \mid M \in \Lambda^\varepsilon h(g(x)), N \in \Lambda^\varepsilon g(x) \}$$

*define an unbounded derivate container for* $\phi = h \circ g$.

## 3. Remarks and Examples

3.1 *Remark.* Let $f\colon V \to \mathbb{R}^n$ be continuous and $(f_l)$ any sequence of $C^1$ functions converging uniformly to $f$ in some neighborhood $U$ of $\bar{x}$. If $f$ is not one-to-one in all sufficiently small neighborhoods of $\bar{x}$ then

$$\limsup_{l\to\infty,\, y\to\bar{x}} |f_l'(y)^{-1}| = \infty.$$

This is a direct consequence of Theorem 2 and the comments following the statement of Theorem 2.

3.2 *Remark.* Let $f \in \mathscr{F}$ and $\operatorname{meas}(N) = 0$, and assume that $f'(\bar{x})$ exists. Then

$$f'(\bar{x}) \in \bigcap_{\varepsilon>0} \overline{\operatorname{co}}\{f'(y) \mid |y-\bar{x}| \leqslant \varepsilon,\, y \notin N\}.$$

This follows directly from the first statement of Theorem 4.

3.3 Example. We shall now construct a Lipschitzian function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ and a derivate container $\Lambda^\varepsilon f$ such that $\partial f(0)$ contains the null matrix but $\Lambda f(0)$ has only nonsingular elements. The function $f$ is "piecewise" linear and a crude approximation to the function $g$ which maps points with polar coordinates $(r, \theta)$ into $(r, 3\theta)$ for $0 \leqslant \theta \leqslant \pi/2$ and into $(r, \theta/3 + (4/3)\pi)$ for $(\pi/2) \leqslant \theta \leqslant 2\pi$. [The function $g$ was constructed by Blank [1] as an example of a one-to-one fat Lipschitzian function with $\partial g(0)$ containing a singular element.] Specifically, if $\theta$ is the polar angle of $x \in \mathbb{R}^2$ then we set

$$f(x) = M_j x \text{ for } \frac{\pi}{6} j \leqslant \theta \leqslant \frac{\pi}{6}(j+1) \qquad (j = 0,\ldots, 11)$$

with

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 3 & -2\sqrt{3} \\ 2\sqrt{3} & -3 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}, \qquad M_3 = \begin{pmatrix} -1/3 & 0 \\ 0 & -1 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} -1/3 & 0 \\ \sqrt{3}/3 & -2/3 \end{pmatrix}, \qquad M_5 = M_6 = \begin{pmatrix} -1/2 & -\sqrt{3}/6 \\ \sqrt{3}/2 & -1/6 \end{pmatrix},$$

$$M_7 = \begin{pmatrix} -1/3 & -\sqrt{3}/3 \\ \sqrt{3}/3 & 1/3 \end{pmatrix}, \qquad M_8 = M_9 = \begin{pmatrix} 1/6 & -\sqrt{3}/2 \\ \sqrt{3}/6 & 1/2 \end{pmatrix},$$

$$M_{10} = \begin{pmatrix} 2/3 & -\sqrt{3}/3 \\ 0 & 1/3 \end{pmatrix}, \qquad M_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix}.$$

We have $0 = \frac{1}{4}M_0 + \frac{3}{4}M_3 \in \partial^\varepsilon f(0)$ for all $\varepsilon > 0$; hence $0 \in \partial f(0)$. We construct $\Lambda^\varepsilon f(x)$ as follows: we choose two sequences $(a_i)$ and $(b_i)$ of positive numbers such that $a_i \leqslant b_i^2$ and $R_i = [-a_i, a_i] \times [-b_i, b_i] \subset \bar{B}(0, 1/i)$, denote by $\chi_i$ the characteristic function of the thin upright rectangle $R_i$, and for each $x \in \mathbb{R}^2$ and $i = 1, 2, \ldots$, set

$$p_i(x) = (4a_i b_i)^{-1} \chi_i(x), \qquad f_i(x) = \int f(x+z)\, p_i(z)\, dz,$$

$$\Lambda^\varepsilon f(x) = \mathrm{cl}\{ f_i'(y) \mid |y - x| \leqslant \varepsilon,\ 1/i \geqslant \varepsilon \}.$$

Then, by Theorem 3, $\Lambda^\varepsilon f$ is an unbounded derivate container for $f$ which is also a derivate container because $\Lambda^\varepsilon f(x)$ are closed and uniformly bounded. Furthermore,

$$f_i'(y) = \int f'(y+z)\, p_i(z)\, dz,$$

and thus $f_i'(y)$ is the area average of $f'$ over the rectangle $y + R_i$. Straightforward (but tedious) computations, carried out separately for the various locations of $y + R_i$ relative to the angles determined by the lines $\theta_j = (\pi/6)j$, show that the determinants of all $f_i'(y)$ are uniformly bounded away from 0 for large $i$. Since $|f_i'(y)|$ are uniformly bounded (by $\max_j |M_j|$), this shows that $|f_i'(y)^{-1}|$ are uniformly bounded for large $i$ and thus $\Lambda f(x)$ contain no singular elements and Theorems 1 and 2 are applicable.

3.4 Example. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \big| |x| + y \big| + \frac{1}{2}x$. By using a chain rule for derivate containers (similar to Theorem 5) we have constructed in [6, 2.13, pp. 17, 18] a derivate container $\Lambda^\varepsilon f$ such that $(0, 0)^T \notin \Lambda f(0, 0)$ which shows [6, Theorem 2.9, p. 17] that $(0, 0)$ does not minimize $f$. We now observe that the same $\Lambda^\varepsilon f$ can be obtained by the use of convolutions, with $p_i$ defined as in Example 3.3.

3.5 *Remark.* We shall verify in Section 4 that the following proposition is valid:

Theorem 6. *If $f\colon V \to \mathbb{R}^n$ is fat at $\bar{x}$ and $f'(\bar{x})$ exists then $f'(\bar{x})$ is nonsingular.*

It follows from the classical implicit function theorem and Theorem 6 that a $C^1$ function $f\colon V \to \mathbb{R}^n$ is fat at $\bar{x}$ if and only if $f'(\bar{x})$ is nonsingular. If $f\colon V \to \mathbb{R}^n$ has a derivate container $\Lambda^\varepsilon f$ such that $\Lambda f(\bar{x})$ is free of singular elements then, by Theorem 2, $f$ is fat at $\bar{x}$ and locally one-to-one. It remains for us an open question whether the converse is also true, that is, whether a locally one-to-one Lipschitzian function $f: V \to \mathbb{R}^n$ that is fat at $\bar{x}$ must admit a derivate container $\Lambda^\varepsilon f$ such that $\Lambda f(\bar{x})$ has no singular elements. This is closely related to the following question: If $f$ is Lipschitzian, one-to-one and fat in some neighborhood of $\bar{x}$, does there exist a sequence $(f_i)$ of $C^1$ functions converging to $f$ uniformly in some neighborhood of $\bar{x}$ and with $|f_i'(x)^{-1}|$ uniformly bounded in that neighborhood? What if $f$ is continuous but not necessarily Lipschitzian?

3.6 Example. The following simple example illustrates an application of Theorems 1 and 4. Let $f = (f^1, f^2): \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f^1(x, y) = 3(x + y)^{1/3}, \qquad f^2(x, y) = x + y^2.$$

Then

$$f'(x, y) = \begin{pmatrix} (x+y)^{-2/3} & (x+y)^{-2/3} \\ 1 & 2y \end{pmatrix} \qquad (x + y \neq 0)$$

and, for $|(x, y)|$ defined as $|x| + |y|$ and $|(x, y)| \leqslant \frac{1}{4}$,

$$\partial^\varepsilon f(x, y) \subset \left\{ \begin{pmatrix} a & a \\ 1 & b \end{pmatrix} \,\middle|\, a \geqslant 4^{2/3} + o(1), |b| \leqslant 1/2 + o(1) \right\},$$

where $o(1)$ denotes a quantity converging to 0 as $\varepsilon \to 0$. We have

$$\begin{pmatrix} a & a \\ 1 & b \end{pmatrix}^{-1} = [a(b-1)]^{-1} \begin{pmatrix} b & -a \\ -1 & a \end{pmatrix}$$

and the norms of the above matrices are bounded by $4 + o(1)$. Thus, by Theorems 1 and 4, $f$ is fat at $(0, 0)$ and a local homeomorphism onto $\bar{B}(0, 1/16)$, with $f(\bar{B}(0, a)) \supset \bar{B}(0, a/4)$ for $0 \leqslant a \leqslant 1/4$.

3.7 *Remark.* If $f: V \to \mathbb{R}^m$ is $C^1$ then $\partial f(\bar{x}) = \{f'(\bar{x})\}$ and thus, by Theorem 4, $f'(\bar{x}) \in \text{co}\, \Lambda f(\bar{x})$ for every derivate container $\Lambda^\varepsilon f$. It may be, however, that $f'(\bar{x}) \notin \Lambda f(\bar{x})$. Consider, e.g., $f = (f^1, f^2): \mathbb{R} \to \mathbb{R}^2$ and $f_i = (f_i^1, f_i^2): \mathbb{R} \to \mathbb{R}^2$, defined by

$$f^1(x) = 0,\ f^2(x) = 0,\ f_i^1(x) = \frac{1}{i} \cos ix,\ f_i^2(x) = \frac{1}{i} \sin ix.$$

Then $\lim_i f_i = f$ uniformly and each $f_i'(y) = (-\sin iy, \cos iy)$ belongs to the boundary of the unit circle. Thus

$$(0, 0) \notin \Lambda f(x) = \bigcap_{\varepsilon > 0} \text{cl}\{f_i'(y) \mid |y - x| \leqslant \varepsilon, i \geqslant 1/\varepsilon\}.$$

A similar example is provided by the functions $g$, $g_i$: $\mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$g(x_1, x_2) = f(x_1 + x_2) = 0, \qquad g_i(x_1, x_2) = f_i(x_1 + x_2).$$

## 4. Proofs

If $\Lambda^\epsilon f$ is an unbounded derivate container for $f$: $V \to \mathbb{R}^m$ then for every compact $V^* \subset V$ there exist a sequence $(f_i)$ and for every $\varepsilon_0 > 0$ numbers $i_0 = i(\varepsilon_0, V^*)$ and $\delta_0 = \delta(\varepsilon_0, V^*)$ satisfying the conditions of Definition A. We shall say, for brevity, that $[(f_i), i_0, \delta_0]$ corresponds to $[\Lambda^\epsilon f, V^*, \varepsilon_0]$. It is clear that $i_0$ may be replaced by any larger number and $\delta_0$ by any smaller number.

We shall denote by $B(A, r)$, respectively, $\bar{B}(A, r)$ the open, respectively, closed $r$-neighborhoods of a set $A$.

4.1 Lemma. *Let $\Lambda^\epsilon f$ be a derivate container for $f$: $V \to \mathbb{R}^n$, $W$ a compact subset of $V$,*

$$\gamma = \sup\{|M^{-1}| \mid M \in \Lambda f(x), x \in W\} < \infty,$$

*and $[(f_i], i(\varepsilon), \delta(\varepsilon)]$ correspond to $[\Lambda^\epsilon f, W, \varepsilon]$. Then for every $\eta > 0$ there exists $i_1 > 0$ such that*

$$|f_i'(y)^{-1}| \leqslant \gamma + \eta \qquad (y \in W, i \geqslant i_1).$$

*Proof.* Assume the conclusion invalid. Then there exist $\eta_0 > 0$ and sequences $(j_1, j_2, \ldots)$ increasing to $\infty$ and $(y_1, y_2, \ldots)$ in $W$ such that

$$|f_{j_i}'(y_i)^{-1}| > \gamma + \eta_0 \qquad (i = 1, 2, \ldots). \tag{1}$$

Since $W$ is compact, we may assume that $(y_i)$ converges to some $\bar{y} \in W$. Since $|M^{-1}| \leqslant \gamma$ for all $M$ in the compact set $\Lambda f(\bar{y})$, there exist $a, b > 0$ such that $|M^{-1}| \leqslant \gamma + \eta_0/2$ for $M \in \Lambda^a f(\bar{y}) \subset B(\Lambda f(\bar{y}), b)$. It follows that

$$|f_i'(y)^{-1}| \leqslant \gamma + \eta_0/2 \qquad \text{if} \quad i \geqslant i(a) \text{ and } |y - \bar{y}| \leqslant \delta(a),$$

thus contradicting (1). Q.E.D.

4.2 *Proof of Theorem* 1

*Step* 1. Let $\eta > 0$, $\beta_1 = \beta + \eta$, $V^* = \bar{B}(\bar{x}, \alpha + 2\eta)$ and $(f_i)$ be the sequence corresponding to $V^*$ as in Definition A. We shall first show that there exists $i_1$ such that

$$f_i'(y)^{-1} \leqslant \beta_1 \qquad (i \geqslant i_1, y \in V^*). \tag{1}$$

Indeed, otherwise there exist sequences $(k_1, k_2,...)$ increasing to $\infty$ and $(y_j)$ in $V^*$ such that

$$|f_{k_j}(y_j)^{-1}| > \beta_1 \qquad (j = 1, 2,...), \tag{2}$$

and we may assume that $(y_j)$ converges to some $\bar{y}$ in the compact set $V^*$. We have

$$\inf_{\epsilon>0} \sup\{|M^{-1}| \mid M \in \Lambda^\epsilon f(\bar{y})\} \leqslant \beta$$

and therefore there exists $\varepsilon_1 > 0$ such that

$$|M^{-1}| \leqslant \beta_1 \qquad (M \in \Lambda^{\epsilon_1} f(\bar{y})).$$

It follows then from Definition A that there exist $i^*$ and $\delta^*$ such that

$$|f_i'(y)^{-1}| \leqslant \beta_1 \qquad (i \geqslant i^*, |y - \bar{y}| \leqslant \delta^*),$$

contradicting (2). Thus (1) must be valid.

*Step* 2. It follows from (1) and [7, Lemma 1, p. 66] that for each $i \geqslant i_1$ there exists a unique $C^1$ function $u_i$: $\bar{B}(f_i(\bar{x}), (\alpha + \eta)/\beta_1) \to \bar{B}(\bar{x}, \alpha + \eta)$ such that

$$f_i(u_i(y)) = y, \qquad |u_i'(y)| = |f_i'(u_i(y))^{-1}| \leqslant \beta_1 \tag{3}$$

for all $y \in \bar{B}(f_i(\bar{x}), (\alpha + \eta)/\beta_1)$ and

$$u_i(\bar{B}(f_i(\bar{x}), a/\beta_1)) \subset \bar{B}(\bar{x}, a) \qquad (0 \leqslant a \leqslant \alpha + \eta). \tag{4}$$

We choose $i_2 \geqslant i_1$ such that $|f_i(\bar{x}) - f(\bar{x})| < \eta/\beta_1$, hence

$$\bar{B}(f_i(\bar{x}), (\alpha + \eta)/\beta_1) \supset \bar{B}(f(\bar{x}), \alpha/\beta_1)$$

for $i \geqslant i_2$, and set $v_i = u_i \mid \bar{B}(f(\bar{x}), \alpha/\beta_1)$ for $i \geqslant i_2$. Then, by (3), the bounded sequence $(v_i)$ is equicontinuous and therefore there exist $J \subset (1, 2,...)$ and $\bar{u}$: $\bar{B}(f(\bar{x}), \alpha/\beta_1) \to \bar{B}(\bar{x}, \alpha + \eta)$ such that $\lim_{i\in J} v_i = \bar{u}$ uniformly on $\bar{B}(f(\bar{x}), \alpha/\beta_1)$.

We now deduce from (4) that

$$\bar{u}(\bar{B}(f(\bar{x}), a/\beta_1)) \subset \bar{B}(\bar{x}, a) \qquad (0 \leqslant a \leqslant \alpha) \tag{5}$$

and from (3) that

$$f(\bar{u}(y)) = y \qquad (y \in \bar{B}(f(\bar{x}), \alpha/\beta_1)) \tag{6}$$

and that $\beta_1$ is a Lipschitz constant for $\bar{u}$. If we write $\bar{u}_\eta$ for $\bar{u}$ to denote its dependence on $\eta$ then relations (5) and (6) are valid with $\bar{u}$, $\beta_1$ replaced by

$\bar{u}_\eta$, $\beta + \eta$. Our previous argument shows that the sequence $(\bar{u}_{1/j})$ has a subsequence converging uniformly to some $w$. Then relations (5) and (6) remain valid with $\bar{u}, \beta_1$ replaced by $w$, $\beta$ and $w$ has $\beta$ as a Lipschitz constant.

With the above modification, relation (6) shows that the Lipschitzian function $w$ in $\mathbb{R}^n$ is one-to-one and therefore an open mapping. Thus $w$ maps the open ball $B = B(f(\bar{x}), a/\beta)$ onto an open subset $A$ of $\bar{B}(\bar{x}, \alpha)$ and $w$ maps the boundary of $B$ onto the boundary of $A$. Therefore $f|\bar{A}$ is a homeomorphism of $\bar{A}$ onto $\bar{B}(f(\bar{x}), a/\beta)$ with inverse $w$. Relation (5), as modified, now implies that

$$\bar{B}(f(\bar{x}), a/\beta) \subset f(\bar{B}(\bar{x}, a)) \qquad (0 \leqslant a \leqslant \alpha).$$

Finally, assume that $\Lambda^\varepsilon f$ is a derivate container for $f$, and let $\eta > 0$ be such that $V^* = B(\bar{x}, \alpha + 2\eta) \subset V$. Then it follows from Lemma 4.1, for $W = V^*$, that relation (1) remains valid with $\beta_1$ replaced by $\gamma_1 = \gamma + \eta$, and the arguments of Step 2 yield the last conclusion of the theorem. Q.E.D.

4.3 *Proof of Theorem* 2. Let $\beta > \beta_0$. Then there exists $\varepsilon_0 > 0$ such that

$$\bar{B}(\bar{x}, \varepsilon_0) \subset V, \qquad |M^{-1}| \leqslant \beta \quad (M \in \Lambda^{\varepsilon_0} f(\bar{x}))$$

Now let $[(f_i), i_0, 3\delta_0]$ correspond to $[\Lambda^\varepsilon f, \bar{B}(\bar{x}, \varepsilon_0), \varepsilon_0]$, with $3\delta_0 \leqslant \varepsilon_0$, and set

$$\Lambda_1^\varepsilon f(x) = \{f_i'(y) \mid |y - x| \leqslant \delta_0, i \geqslant \max(1/\varepsilon, i_0)\}$$

for $\varepsilon > 0$ and $x \in \bar{B}(\bar{x}, 2\delta_0)$. Then $\Lambda_1^\varepsilon f$ is an unbounded derivate container for $f| B(\bar{x}, 2\delta_0)$ and the conclusion of the theorem follows from Theorem 1 with $V, f, \Lambda^\varepsilon f, \alpha$ replaced by $B(\bar{x}, 2\delta_0), f| B(\bar{x}, 2\delta_0), \Lambda_1^\varepsilon f, \delta_0$, respectively.

If $\Lambda^\varepsilon f$ is a derivate container for $f$ then it is easy to verify that $\gamma_0 = \beta_0$ because of the compactness of $\Lambda^\varepsilon f(\bar{x})$. Q.E.D.

4.4 *Proof of Theorem* 3. Let $Y = \mathbb{R}^n$ or $Y = \mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$, $i \in \{1, 2, \ldots\}$ and $g: V \to Y$ be locally integrable. Set

$$\gamma_i(x) = \int g(x + z)\, p_i(z)\, dz \qquad (x \in V).$$

Then $\lim_{h \to 0} \int_P |g(y + h) - g(y)|\, dy = 0$ for every bounded measurable set $P$ and therefore, for each $x \in V$ and $B = B(x, 1/i)$,

$$\begin{aligned} \lim_{h \to 0} |\gamma_i(x + h) - \gamma_i(x)| &= \lim_{h \to 0} \left| \int [g(x + h + z) - g(x + z)]\, p_i(z)\, dz \right| \\ &\leqslant |p_i|_\infty \lim_{h \to 0} \int_B |g(y + h) - g(y)|\, dy = 0. \end{aligned}$$

Thus $\gamma_i$ is continuous.

Now let $F_i(x) = \int f'(x+z)\, p_i(z)\, dz$ ($x \in V$). Each $F_i(x)$ is defined because $f'$ is dominated by $\psi_f$. Furthermore, our previous argument shows that both $f_i$ and $F_i$ are continuous. For each $x \in V$ and $h \in \mathbb{R}^n$, with $\bar{B}(x, 1/i) \subset V$, $\bar{B}(x+h, 1/i) \subset V$ and $|h| \leqslant a_f$, we have

$$|h|^{-1}\, |f_i(x+h) - f_i(x) - F_i(x)h|$$

$$\leqslant |p_i|_\infty \int_{B(0,1/i)} |h|^{-1}\, |f(x+h+z) - f(x+z) - f'(x+z)h|\, dz.$$

The integrand in the last expression converges to 0 as $h \to 0$ for almost all $z$ and it is bounded by $2\psi_f(x+z)$. Thus the integral on the right converges to 0 as $h \to 0$, showing that

$$f_i'(x) = F_i(x) \qquad (\bar{B}(x, 1/i) \subset V)$$

Now let $V^* \subset V$ be compact. Then $f$ is uniformly continuous on $V^*$. Let $\varepsilon > 0$ and $\delta = \delta(\varepsilon) > 0$ be such that

$$|f(x_1) - f(x)| \leqslant \varepsilon \qquad \text{if} \quad |x_1 - x| \leqslant \delta, \quad x \in V^*.$$

For each $x \in V^*$ and $i \geqslant 1/\delta$, with $\bar{B}(V^*, 1/i) \subset V$, we have

$$|f_i(x) - f(x)| = \left| \int_{B(0,1/i)} [f(x+z) - f(x)]\, p_i(z)\, dz \right|$$
$$\leqslant \varepsilon\, |p_i|_1 \leqslant c\varepsilon.$$

Thus $\lim_i f_i = f$ uniformly on $V^*$.

Finally, we verify that $\Lambda^\varepsilon f$ satisfies Definition A, with $(f_i)$ as defined above for each compact $V^* \subset V$ and with $i(\varepsilon, V^*) = 1/\varepsilon$, $\delta(\varepsilon, V^*) = \varepsilon$.

Q.E.D.

We shall prove Theorems 4 and 5 in reverse order.

4.5 *Proof of Theorem* 5

It is clear that $\Lambda^\varepsilon \phi(x) \subset \Lambda^{\varepsilon'} \phi(x)$ if $\varepsilon < \varepsilon'$. Now let $V_1^* \subset V_1$ be compact, $V_2^* = g(V_1^*)$, and $[(g_i), \delta_1(\varepsilon), i_1(\varepsilon)]$, respectively, $[(h_i), \delta_2(\varepsilon), i_2(\varepsilon)]$ correspond to $[\Lambda^\varepsilon g, V_1^*, \varepsilon]$, respectively, $[\Lambda^\varepsilon h, V_2^*, \varepsilon]$. We may assume that $g_i(V_1^*) \subset V_2$ for all $i$, otherwise replacing $(g_i)$ by $(g_i)_{i > i_0}$ for some appropriate $i_0$. Then $\phi_i = h_i \circ g_i$ are $C^1$ in some neighborhood $\tilde{V}_1$ of $V_1^*$ and $\lim_i \phi_i = \phi$ uniformly on $V_1^*$.

For each $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$ and $i(\varepsilon) = \max(i_1(\varepsilon), i_2(\varepsilon))$ such that

$$|g_i(y) - g(x)| \leqslant \delta_2(\varepsilon) \qquad \text{and} \qquad |y - x| \leqslant \delta_1(\varepsilon)$$

if $|y-x| \leqslant \delta(x)$ and $i \geqslant i(\varepsilon)$. Thus, for $i \geqslant i(\varepsilon)$, $y \in V_1$, $x \in V_1^*$ and $|y-x| \leqslant \delta(\varepsilon)$, we have

$$\phi_i'(y) = h_i'(g_i(y))\, g_i'(y) \in \Lambda^\varepsilon \phi(x).$$

Q.E.D.

In order to prove Theorem 4 we first require a lemma. The proof of this lemma is patterned after the proof of [6, Theorem 2.7, p. 25].

4.6 Lemma. *Let $n = m = 1$, $f \in \mathscr{F}$, and $\Lambda^\varepsilon f$ be an unbounded derivate container for $f$. Then for each compact $V^* \subset V$ there exist numbers $\delta(\varepsilon) > 0$ $(\varepsilon > 0)$ such that*

$$\partial^{\delta(\varepsilon)} f(x) \subset \mathrm{cl}[\Lambda^\varepsilon f(x)] \qquad (\varepsilon > 0, x \in V^*).$$

*Proof.* Since $V^*$ is compact and $V$ open, there exists $\omega > 0$ such that $B(V^*, 2\omega) \subset V$. Let $\varepsilon > 0$ and $[(f_i), i_0, 2\delta]$ correspond to $[\Lambda^\varepsilon f, \bar{B}(V^*, \omega), \varepsilon]$, with $2\delta < \omega$. Next consider any $x \in V^*$ and any two points $v_1, v_2 \in \bar{B}(x, \delta)$ such that $f'(v_1)$ and $f'(v_2)$ exist. For any $\eta > 0$ there exists $h$ such that $0 < h < \delta$ and

$$|f'(v_k) - h^{-1}[f(v_k + h) - f(v_k)]| < \eta \qquad (k = 1, 2). \tag{1}$$

We may choose $i_1$ large enough so that $i_1 \geqslant i_0$ and

$$|h^{-1}[f(v_k + h) - f(v_k)] - h^{-1}[f_i(v_k + h) - f_i(v_k)]| < \eta$$
$$(i \geqslant i_1, k = 1, 2). \tag{2}$$

By the mean-value theorem, there exist $\theta_{ki} \in [0, 1]$ such that

$$h^{-1}[f_i(v_k + h) - f_i(v_k)] = f_i'(v_k + \theta_{ki} h) \qquad (k = 1, 2, i \geqslant i_1). \tag{3}$$

Since $f_i'$ is continuous, the closed interval $S_i$ joining the points $f_i'(v_1 + \theta_{1i} h)$ and $f_i'(v_2 + \theta_{2i} h)$ is contained in $f_i'(I_i)$, where $I_i$ denotes the closed interval joining $v_1 + \theta_{1i} h$ and $v_2 + \theta_{2i} h$. Since $I_i \subset [x - 2\delta, x + 2\delta]$, we have

$$S_i \subset f_i'(I_i) \subset \Lambda^\varepsilon f(x).$$

Thus, by (1)–(3), the closed interval $J$ joining $f'(v_1)$ and $f'(v_2)$ is contained in $\bar{B}(\Lambda^\varepsilon f(x), 2\eta)$ for every $\eta > 0$; hence $J \subset \mathrm{cl}[\Lambda^\varepsilon f(x)]$. This shows that

$$\partial^\delta f(x) = \overline{\mathrm{co}}\{f'(y) \mid |y - x| \leqslant \delta\} \subset \mathrm{cl}[\Lambda^\varepsilon f(x)]$$

if $x \in V^*$ and $\varepsilon > 0$. Q.E.D.

4.7 *Proof of Theorem* 4

*Step* 1. Let $f \in \mathscr{F}$ and meas$(N) = 0$. For each $i = 1, 2, \ldots$ let

$$p_i(x) = 1/\text{meas}(B(0, 1/i)) \qquad (|x| \leqslant 1/i), \quad p_i(x) = 0 \quad (|x| > 1/i),$$

$$f_i(x) = \int f(x+z)\, p_i(z)\, dz \qquad (\bar{B}(x, 1/i) \subset V),$$

$$\Lambda_1^\varepsilon f(x) = \{f_i'(y) \mid |y - x| \leqslant \varepsilon, \bar{B}(y, 1/i) \subset V, i \geqslant 1/\varepsilon\}.$$

Then, by Theorem 3, $\Lambda_1^\varepsilon f$ is an unbounded derivate container for $f$. Furthermore, since each $\partial_N^\varepsilon f(x)$ is closed and convex and each $p_i$ a probability density function, we have (by Theorem 3),

$$f_i'(y) = \int f'(y+z)\, p_i(z)\, dz = \int_{z \notin N - y} f'(y+z)\, p_i(z)\, dz \in \partial_N^{2\varepsilon} f(x)$$

if $|y - x| \leqslant \varepsilon$, $\bar{B}(y, 1/i) \subset V$, and $1/i \leqslant \varepsilon$. Thus

$$\Lambda_1^\varepsilon f(x) \subset \partial_N^{2\varepsilon} f(x) \qquad (\varepsilon > 0, x \in V)$$

which shows that $\partial_N^\varepsilon f$ is also an unbounded derivate container for $f$.

*Step* 2. Now let $\Lambda^\varepsilon f$ be any unbounded derivate container for $f$ and $V^*$ a compact subset of $V$. For any $x \in V$ and arbitrary $a_j \in \mathbb{R}^m$, $b_j \in \mathbb{R}^n$ with $|b_j| = 1$ $(j = 1, \ldots, n)$, let $\alpha > 0$ be such that $x + tb_j \in V$ $(j = 1, \ldots, n)$ if $|t| < \alpha$. We set

$$\psi(t) = \sum_{j=1}^n a_j^T f(x + tb_j) \qquad (|t| < \alpha)$$

and observe that, by Theorem 5, the sets $\sum_{j=1}^n a_j^T \Lambda^\varepsilon f(x + tb_j)\, b_j$ define an unbounded derivate container for $\psi$. We also verify that

$$\sum_{j=1}^n a_j^T \partial^\varepsilon f(x + tb_j)\, b_j \subset \partial^\varepsilon \psi(t)$$

Thus, by Lemma 4.6, there exist positive numbers $\delta(\varepsilon) > 0$ $(\varepsilon > 0)$ such that

$$\sum_{j=1}^n a_j^T \partial^{\delta(\varepsilon)} f(x)\, b_j \subset \sum_{j=1}^n a_j^T [\text{cl}\, \Lambda^\varepsilon f(x)]\, b_j \subset \sum_{j=1}^n a_j^T\, \overline{\text{co}}\, \Lambda^\varepsilon f(x)\, b_j \tag{1}$$

if $x \in V^*$.

Now let $A$ be an arbitrary $m \times n$ matrix, and let $a_j$ be its $j$th column. Then $A = \sum_{j=1}^n a_j b_j^T$, where $b_j^T = (b_j^1, \ldots, b_j^n)$ is a row vector and

$$b_j^k = 0 \qquad (k \neq j), \qquad b_j^j = 1.$$

We may assume that $|b_j| = 1$, otherwise multiplying $a_j$ and dividing $b_j$ by $|b_j|$. Thus (1) yields

$$A \odot \partial^{\delta(\epsilon)} f(x) \subset A \odot \overline{\text{co}}\, \Lambda^\epsilon f(x) \tag{2}$$

for every $x \in V^*$ and every $m \times n$ matrix $A$, where $\odot$ denotes the scalar product of two $m \times n$ matrices viewed as elements of $\mathbb{R}^{mn}$. Since $\partial^{\delta(\epsilon)} f(x)$ and $\overline{\text{co}}\, \Lambda^\epsilon f(x)$ are, for each $x \in V^*$, closed convex subsets of $\mathbb{R}^{mn}$, the standard theorem about the separation of convex sets implies that

$$\partial^{\delta(\epsilon)} f(x) \subset \overline{\text{co}}\, \Lambda^\epsilon f(x). \tag{3}$$

*Step* 3. Let $y \in V$ be such that $f'(y)$ exists, and set $V^* = \{y\}$. Since $\partial_N^\epsilon f$ is an unbounded derivate container for $f$, relation (3) implies that for each $\eta > 0$ there exists $\delta > 0$ such that

$$f'(y) \in \partial^\delta f(y) \subset \partial_N^\eta f(y).$$

This shows that for each choice of $x \in V$ and $\eta, \varepsilon > 0$ we have

$$\{f'(y) \mid |y - x| \leqslant \varepsilon\} \subset \overline{\text{co}}\{f'(z) \mid |x - z| \leqslant \varepsilon + \eta, z \notin N\}$$

whence we conclude that

$$\partial^\epsilon f(x) \subset \partial_N^{\epsilon'} f(x) \subset \partial^{\epsilon'} f(x) \qquad (\varepsilon' > \varepsilon > 0, x \in V). \tag{4}$$

*Step* 4. Now assume that $\Lambda^\epsilon f$ is a derivate container for $f$ (which implies that $f$ must be Lipschitzian). Then $\partial_N(f)$ and $\Lambda f(x)$ are compact and nonempty because each is the intersection of nested compact sets. It follows then from (4) that $\partial_N f(x) = \partial f(x)$ $(x \in V)$ and from (3) that

$$\partial f(x) \subset \bigcap_{\epsilon > 0} \text{co}\, \Lambda^\epsilon f(x) = \text{co}\, \Lambda f(x).$$

Q.E.D.

4.8 *Proof of Theorem* 6

Assume that $f'(\bar{x})$ exists and $f$ is fat at $\bar{x}$. Then there exist $c > 0$ and points $u(y)$ defined for all $y$ sufficiently close to $f(\bar{x})$ such that

$$f(u(y)) = y \tag{1}$$

and

$$|u(y) - \bar{x}| \leqslant |y - f(\bar{x})|/c. \tag{2}$$

Let $z \in \mathbb{R}^n$, $|z| = 1$, $(t_i)$ be a sequence of positive numbers decreasing to 0,

$$y_i = f(\bar{x}) + t_i z, \qquad u_i = u(y_i).$$

By (2), $|u_i - \bar{x}|/t_i \leqslant 1/c$. Thus we may assume that $p = \lim_i t_i^{-1}(u_i - \bar{x})$ exists (otherwise replacing $(t_i)$ by an appropriate subsequence). We have, by (1),

$$\begin{aligned} 0 &= \lim_i |u_i - \bar{x}|^{-1} |f(u_i) - f(\bar{x}) - f'(\bar{x})(u_i - \bar{x})| \\ &= \lim_i t_i |u_i - \bar{x}|^{-1} |z - f'(\bar{x})(u_i - \bar{x})/t_i| \\ &\geqslant c |z - f'(x) p|. \end{aligned}$$

Thus, for every $z \in \mathbb{R}^n$ with $|z| = 1$, the equation $f'(\bar{x}) p = z$ has a solution $p$, showing that $f'(\bar{x})$ is nonsingular. Q.E.D.

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